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On the Anderson-Livingston Graphs of 4-Radical Index of Nilpotence Finite Completely Primary Rings

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Abstract

The classification of zero-divisor graphs of finite commutative rings is an active research area with connections to algebraic combinatorics. In this paper, we investigate the Anderson-Livingston graphs $\Gamma(R)$ associated with completely primary finite rings R whose maximal ideal $Z(R)$ satisfies $(Z(R))^4 = (0)$ but $(Z(R))^3 \neq (0)$. Using the Raghavendran idealization procedure, we construct and characterize four classes of such rings with characteristics p, p^2, p^3, and p^4, explicitly describing the zero-divisor structure and radical filtration. For each class, we determine the basic graph invariants—diameter, girth, clique number, chromatic number, and degree sequences—proving that characteristics p, p^2, and p^3 yield incomplete graphs of diameter 2, while characteristic p^4 gives a complete graph. Additionally, we compute several advanced invariants: binding number, Zagreb indices, Wiener index, Mostar index, metric dimension, fractional metric dimension, edge differential, and domination number for the concrete examples with $p = 2, r = 1$, which are illustrated with TikZ drawings. We further provide a complete classification of the automorphism groups $\mathrm{Aut}(\Gamma(R))$ as products of symmetric groups, deriving explicit orbit decompositions. Our results extend the classification of zero-divisor graphs for higher radical indices and highlight the deep interplay between algebraic structure and graph theory.

Keywords: Zero Divisor Graph, Completely Primary Ring, Finite Ring, Anderson-Livingston Graph, Automorphism Group, Galois Ring, Idealization

MSC (2020): 05C25, 16U99, 16P10, 13M99

1 Introduction

The study of zero-divisor graphs of commutative rings originated with Beck [5], who introduced the concept of a zero-divisor graph by considering all ring elements as vertices, with two vertices adjacent if and only if their product is zero in the ring. Beck was primarily interested in the coloring problem, conjecturing that the chromatic number of such a graph equals its clique number, a conjecture that was later shown to be false by Anderson and Naseer [2]. The original definition was subsequently refined by Anderson and Livingston [1], who restricted the vertex set to the nonzero zero-divisors of the ring, producing the graph $\Gamma(R)$ that has since become the standard object of study in the field. This refinement was motivated by the desire to give a clearer illustration of the zero-divisor structure of the ring, and it has proven to be a fruitful framework for investigating the interplay between ring-theoretic and graph-theoretic properties. Since the publication of Anderson and Livingston's foundational work, the zero-divisor graph has been extensively studied for various classes of rings, with researchers exploring graph invariants such as diameter, girth, clique number, chromatic number, planarity, and the automorphism group, among others.

The classification problem for finite rings provides a fundamental motivation for the study of zero-divisor graphs. By the structure theorem of Raghavendran [15], every finite ring with identity decomposes as a direct sum of matrix rings over completely primary finite rings. Consequently, understanding the structure of completely primary finite rings—rings whose zero divisors form a unique maximal ideal—is essential for the broader classification of finite rings. Completely primary finite rings have attracted considerable attention due to their role as building blocks in the classification of finite rings; indeed, a finite ring has a unique maximal ideal if and only if it is a full matrix ring over a completely primary finite ring, and any commutative finite ring is a direct sum of completely primary finite rings. The Galois rings, which are the trivial examples of completely primary finite rings, and their idealizations have been studied extensively for the structures of their unit groups and zero-divisor graphs up to isomorphism. The unit groups of completely primary finite rings have been well characterized in the literature (see, for instance, the work of Ojiema, Oduor, and Oleche [13] and Oduor, Ojiema and Mmasi [12]), yet the zero-divisor compartment—which lacks the algebraic closure properties of the unit group—has received comparatively less attention.

Within the framework of zero-divisor graphs, a substantial body of research has focused on determining graph invariants for specific classes of rings and on characterizing rings whose zero-divisor graphs satisfy certain properties. Anderson, Frazier, Lauve, and Livingston [3] studied the clique number of zero-divisor graphs and the relationship between graph isomorphisms and ring isomorphisms, and they determined all n for which $\Gamma(\mathbb{Z}_n)$ is planar. Subsequent work by Akbari, Maimani, and Yassemi [4], and independently by Smith [16], provided a complete classification of finite rings with planar zero-divisor graphs, listing forty-four isomorphism classes. Liu, Wu, and Guo [8] extended the characterization of finite commutative rings whose zero-divisor graphs have clique number two or three to the case of clique number four. More recently, researchers have turned their attention to the zero-divisor graphs of completely primary finite rings with prescribed radical indices. For instance, Ndago, Oduor, and Ojiema [10] investigated the adjacency, Laplacian, and distance matrices of zero-divisor graphs of 3-radical zero completely primary finite rings, while Were and Oduor [17] provided a characterization of zero-divisor graphs of completely primary finite rings satisfying $(Z(R))^5 = (0)$ and $(Z(R))^4 \neq (0)$. Despite these advances, the classification of zero-divisor graphs for rings with higher radical indices remains incomplete, and a systematic treatment of the 4-radical index case—where $(Z(R))^4 = (0)$ but $(Z(R))^3 \neq (0)$—has been lacking. The study of automorphism groups of zero-divisor graphs represents another active and challenging direction within the field. Understanding the symmetries of $\Gamma(R)$ provides deep insights into the algebraic structure of the underlying ring. Ou, Wang, and Tian [14] described the automorphism group of the zero-divisor graph of a finite semisimple ring and of a block upper triangular matrix ring over a finite field. For completely primary finite rings, the automorphism groups of zero-divisor graphs have been investigated for rings with square radical zero and cube radical zero. However, the automorphism groups for rings with 4-radical index of nilpotence have not been fully classified, and the structure of the associated zero-divisor graphs for all four possible characteristics (p, p^2, p^3, and p^4) remains an open problem. The present paper addresses this gap by providing a complete classification of the automorphism groups of the Anderson-Livingston graphs for completely primary finite rings with $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$ across all four characteristic classes.

The idealization procedure, introduced by Nagata [9], provides a systematic method for constructing rings with prescribed ideals and has been instrumental in the study of completely primary finite rings. Given a base ring R_0 and an R_0-module U, the idealization $R = R_0 \oplus U$ endows the direct sum with a multiplication that encodes the module structure. This construction has been used extensively to build classes of completely primary finite rings with controlled radical filtrations. Chikunji [6] classified cube radical zero completely primary finite rings using this approach, and subsequent work has extended these classifications to higher radical indices. In this paper, we employ the idealization procedure to construct four classes of completely primary finite rings with characteristics p, p^2, p^3, and p^4, all satisfying $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$. For each class, we explicitly describe the zero-divisor structure, the radical filtration, and the resulting Anderson-Livingston graph $\Gamma(R)$.

Our main results provide a complete characterization of the graph invariants for each characteristic class. We prove that rings of characteristics p, p^2, and p^3 yield incomplete graphs of diameter 2, while rings of characteristic p^4 produce complete graphs of diameter 1. For each class, we determine the number of vertices, clique number, chromatic number, and degree sequences. We further provide a complete classification of the automorphism groups $\mathrm{Aut}(\Gamma(R))$, demonstrating that they decompose as products of symmetric groups on the orbits determined by the annihilator structure. This classification extends the known results for lower radical indices and reveals a clear pattern: the symmetry group of the zero-divisor graph is determined entirely by the radical filtration levels of the elements.

In addition to the basic graph invariants, we compute several advanced invariants for the concrete examples with $p = 2$ and $r = 1$, including the binding number, the first and second Zagreb indices, the Wiener index, the Mostar index, the metric dimension, the fractional metric dimension, the edge differential, and the domination number. These computations, which are illustrated with TikZ drawings, provide a comprehensive picture of the graph-theoretic properties of the constructed rings. The results highlight the rich interplay between algebraic structure and graph theory, and they contribute to the ongoing classification of zero-divisor graphs for finite rings.

2 Preliminaries

2.1 Ring Theory

Throughout, R denotes a finite commutative ring with identity $1 \neq 0$. We recall the following standard definitions.

Definition 2.1. A ring R is called *completely primary* if it has a unique maximal ideal. For a finite commutative ring, this maximal ideal is precisely the set of zero-divisors $Z(R)$.
Definition 2.2. A *Galois ring* $GR(p^{kr}, p^k)$ is a finite ring of order p^{kr} and characteristic p^k containing a maximal Galois subring $GR(p^r, p)$ as its residue field.

The following theorem of Raghavendran [15] is fundamental.
Theorem 2.1 (Raghavendran)**.** *Let R be a finite local ring with identity $1 \neq 0$. Then there exist a prime p and positive integers h, r such that:*

(i) $|R| = p^{(h+1)r}$,

(ii) $J(R)$*, the Jacobson radical, is the unique maximal ideal,*

(iii) $|J(R)| = p^{hr}$,

(iv) $|R^*| = p^{hr}(p^{hr} - 1)$.

The *idealization* construction (also called the trivial extension), introduced by Nagata [9], provides a systematic method for constructing rings with prescribed ideals. Given an R_0-module U, the idealization $R = R_0 \oplus U$ has multiplication $(r, u)(s, v) = (rs, rv + su)$.

2.2 Graph Theory

We follow standard graph theory terminology as in Diestel [7].
Definition 2.3. The *Anderson-Livingston zero-divisor graph* $\Gamma(R)$ of a commutative ring R has vertex set $Z(R)^* = Z(R) \setminus \{0\}$, with distinct vertices u, v adjacent if and only if $uv = 0$.

For a graph G, we denote its diameter by $\mathrm{diam}(G)$, girth by $\mathrm{gr}(G)$, clique number by $\omega(G)$, chromatic number by $\chi(G)$, and the degree of a vertex v by $\deg(v)$.

The following basic result is due to Anderson and Livingston [1].
Theorem 2.2. *For any commutative ring R, $\Gamma(R)$ is connected and* $\mathrm{diam}(\Gamma(R)) \leq 3$*. If R is finite, then $\Gamma(R)$ is finite.*

3 Graphs Determined by Divisors of Galois Rings

Before presenting our main constructions, we first analyze the Anderson-Livingston graphs associated with the divisor sets of trivial Galois rings $\mathbb{Z}_{p^n}$. These results provide useful insights.

Let $R = \mathbb{Z}_{p^n}$ and let $D(R) = \{p^a : 1 \leq a \leq n-1\}$ be the set of nontrivial divisors. We consider the graph $\Gamma(D(R))$ with vertex set $D(R)$ and adjacency defined by zero product in $\mathbb{Z}_{p^n}$.
Proposition 3.1. *For $R = \mathbb{Z}_{p^n}$:*

(i) If $n = 1$, $\Gamma(D(R)) = \emptyset$.

(ii) If $n = 2$, $\Gamma(D(R)) \cong K_1$.

(iii) *If* $n = 3$, $\Gamma(D(R)) \cong K_2$.

(iv) *If* $n = 4$, $\Gamma(D(R)) \cong K_{1,2}$.

Proof. For $n = 1$, $\mathbb{Z}_p$ is a field, so there are no nonzero zero-divisors. For $n = 2$, the only zero-divisor is p, giving a single vertex. For $n = 3$, the vertices p and p^2 are adjacent because $p \cdot p^2 = p^3 = 0$, so the graph is K_2. For $n = 4$, the vertices are p, p^2, p^3; p and p^2 are not adjacent (product $p^3 \neq 0$), while p^3 is adjacent to both p and p^2. Hence the graph is $K_{1,2}$. □

Proposition 3.2. *For* $R = \mathbb{Z}_{p^n}$ *with* $n \geq 2$, $\Gamma(D(R))$ *is connected and has* $n - 1$ *vertices. The degree of the vertex* p^a $(1 \leq a \leq n - 1)$ *is given by*

$$\deg(p^a) = \begin{cases} a, & 2a < n, \\ a - 1, & 2a \geq n. \end{cases}$$

In particular, the degree sequence is symmetric.

Proof. The vertex p^a is adjacent to p^b iff $a + b \geq n$. The number of $b \in \{1, \ldots, n-1\}$, $b \neq a$, satisfying this is exactly a if $2a < n$, and $a - 1$ if $2a \geq n$. This follows by counting the integers b in $[n - a, n - 1]$ and excluding $b = a$ when it lies in that interval. □

For rings $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p^2q}$, etc., similar elementary counting yields the following (stated without proof for brevity):

Proposition 3.3. *(i)* *For* $R = \mathbb{Z}_{pq}$, $\Gamma(D(R)) \cong K_2$.

(ii) *For* $R = \mathbb{Z}_{p^2q}$, $\Gamma(D(R)) \cong P_4$ *(path on 4 vertices).*

(iii) *For* $R = \mathbb{Z}_{p^3q}$, $\Gamma(D(R))$ *has 6 vertices and degree sequence* $(1, 1, 2, 3, 3, 4)$.

4 Main Constructions and Results

In this section, we present the constructions and main theorems for rings of characteristics p, p^2, p^3, and p^4 with $(Z(R))^4 = 0$ and $(Z(R))^3 \neq 0$. The proofs are based on the idealization procedure(see [11] for detailed structural proofs)

4.1 Rings of Characteristic p

Construction 4.1. *Let* $R_0 = GR(p^r, p)$. *Define*

$$R = R_0 \oplus R_0u \oplus R_0v \oplus R_0w$$

with multiplication

$$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0,\ r_0s_1 + r_1s_0,\ r_0s_2 + r_2s_0 + r_1s_1,\ r_0s_3 + r_3s_0 + r_1s_2 + r_2s_1).$$

Theorem 4.1. *The ring R is a commutative completely primary finite ring of characteristic p with maximal ideal $Z(R) = R_0u \oplus R_0v \oplus R_0w$ whose radical filtration is given by*

$$(Z(R))^2 = R_0v \oplus R_0w, \quad (Z(R))^3 = R_0w, \quad (Z(R))^4 = (0).$$

Proof. The ring is commutative and finite, with $|R| = p^{4r}$ and $\text{char}(R) = p$. If $r_0 \neq 0$, then $x = (r_0, r_1, r_2, r_3)$ is a unit with inverse given explicitly (see [6]); if $r_0 = 0$, then $x \in R_0u \oplus R_0v \oplus R_0w$ and $x \cdot (0, 0, 1, 0) = 0$, so x is a zero-divisor. Thus $Z(R) = R_0u \oplus R_0v \oplus R_0w$. Direct computation from the multiplication rules yields

$$(Z(R))^2 = R_0v \oplus R_0w, \quad (Z(R))^3 = R_0w, \quad (Z(R))^4 = (0).$$

Any ideal properly containing $Z(R)$ contains a unit, hence is R itself; therefore $Z(R)$ is maximal and R is completely primary. □

Theorem 4.2. *For the above ring R, the Anderson-Livingston graph $\Gamma(R)$ has $p^{3r} - 1$ vertices, is incomplete, has diameter 2, has girth 3 whenever $p^{3r} - 1 \geq 3$, has clique number p^r, and has chromatic number at most $p^{3r} - 2$.*

Proof. Since $|Z(R)| = p^{3r}$, the graph has $p^{3r} - 1$ vertices. It is incomplete because $u \cdot v = w \neq 0$, so u and v are not adjacent. Nevertheless, every pair of vertices has distance at most 2: the element w annihilates every element of $Z(R)$, hence is a universal vertex, so any two vertices are connected through w. Thus $\text{diam}(\Gamma(R)) = 2$. Whenever the graph has at least three vertices, the set $\{u, v, u+v\}$ forms a triangle, as each pair multiplies to 0 in characteristic p; hence the girth is 3 for $p^{3r} - 1 \geq 3$. The largest clique consists of the nonzero multiples of w together with one additional vertex, giving $\omega(\Gamma(R)) = p^r$. Finally, since the maximum degree is at most $p^{3r} - 2$, Brooks' theorem or the greedy coloring bound yields $\chi(\Gamma(R)) \leq p^{3r} - 2$. □

4.2 Rings of Characteristic p^2

Construction 4.2. *Let $R_0 = GR(p^{2r}, p^2)$ and let u_1, u_2 be p^2-torsion with $pu_i \neq 0$, and v be p-torsion. Define*

$$R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$$

with multiplication

$$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0 + p(r_1s_1) + r_1s_2 + r_2s_1 + r_2s_2,$$
$$r_0s_1 + r_1s_0,\ r_0s_2 + r_2s_0,\ r_0s_3 + r_3s_0 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2).$$

Theorem 4.3. *The ring R is commutative completely primary with $\text{char}(R) = p^2$, and*

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v, \quad (Z(R))^2 = pR_0 \oplus pR_0u_1 \oplus pR_0u_2 \oplus R_0v,$$

$$(Z(R))^3 = pR_0u_1 \oplus pR_0u_2, \quad (Z(R))^4 = (0).$$

Theorem 4.4. *For this ring, $\Gamma(R)$ has $p^{4r} - 1$ vertices, is incomplete, has diameter 2, girth 3 (if $p^{4r} - 1 \geq 3$), clique number p^{3r}, and chromatic number at most $p^{4r} - 2$.*

4.3 Rings of Characteristic p^3

Construction 4.3. *Let $R_0 = GR(p^{3r}, p^3)$ and define*

$$R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$$

with $pu_i \neq 0$, $p^3u_i = 0$ for $i = 1, 2$, $pv = 0$, and multiplication

$$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0,\ r_0s_1 + r_1s_0,\ r_0s_2 + r_2s_0,\ r_0s_3 + r_3s_0 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2).$$

Theorem 4.5. *The ring is completely primary with $\mathrm{char}(R) = p^3$, and*

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v, \quad (Z(R))^2 = p^2R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v,$$

$$(Z(R))^3 = pR_0v, \quad (Z(R))^4 = (0).$$

Theorem 4.6. *$\Gamma(R)$ has $p^{5r} - 1$ vertices, is incomplete, has diameter 2, girth 3 (if $p^{5r} - 1 \geq 3$), clique number p^{2r}, and chromatic number at most $p^{5r} - 2$.*

4.4 Rings of Characteristic p^4

Construction 4.4. *Let $R_0 = GR(p^{4r}, p^4)$ and define*

$$R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0u_3$$

with $pu_i = 0$ for $i = 1, 2, 3$, and multiplication

$$(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0,\ r_0s_1 + r_1s_0,\ r_0s_2 + r_2s_0,\ r_0s_3 + r_3s_0).$$

Theorem 4.7. *The ring R is commutative completely primary with $\mathrm{char}(R) = p^4$, and*

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0u_3, \quad (Z(R))^2 = p^2R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0u_3,$$

$$(Z(R))^3 = p^3R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0u_3, \quad (Z(R))^4 = (0).$$

Theorem 4.8. *For the characteristic p^4 ring, $\Gamma(R) \cong K_{p^{4r}-1}$. Hence*

$$|V| = p^{4r} - 1, \quad \mathrm{diam} = 1, \quad \omega = \chi = p^{4r} - 1, \quad \mathrm{gr} = \begin{cases} 3, & p^{4r} - 1 \geq 3, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. Take distinct vertices $x = (0, r_1, r_2, r_3)$ and $y = (0, s_1, s_2, s_3)$ in $Z(R)^*$. Since both have zero first component, the multiplication rule gives $xy = (0, 0, 0, 0)$, so x and y are adjacent. Thus every pair of distinct vertices is adjacent, so $\Gamma(R) \cong K_{p^{4r}-1}$. The invariants follow from the standard properties of complete graphs. □

5 Examples and Graph Properties

In this section we illustrate the main results with concrete examples. For each characteristic class we take $p = 2, r = 1$ so that the Galois ring reduces to $\mathbb{Z}_{2^k}$ and the number of vertices is manageable. We give the explicit ring, the associated Anderson–Livingston graph (drawn with TikZ), and a summary of its algebraic properties.

5.1 Characteristic p: The Graph K_7 minus a Triangle

For $p = 2$, $r = 1$, let $R_0 = \mathbb{Z}_2$ and

$$R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 u \oplus \mathbb{Z}_2 v \oplus \mathbb{Z}_2 w$$

with multiplication as in Construction 4.1. Then $|Z(R)^*| = 2^3 - 1 = 7$. The zero-divisors are the nonzero triples $(a, b, c) \in \mathbb{Z}_2^3$, identified with $au + bv + cw$. The multiplication rules give

$$uv = w, \qquad uw = vw = 0, \qquad u^2 = v^2 = w^2 = 0.$$

Thus two vertices are nonadjacent precisely when their product is w, which occurs exactly for the pairs

$$\{u, v\}, \quad \{u, u + v\}, \quad \{v, u + v\}.$$

Hence $\Gamma(R) \cong K_7$ with the triangle on $\{u, v, u + v\}$ removed.

The graph invariants are:

$$\begin{aligned}
|V| &= 7, \\
\mathrm{diam}(\Gamma) &= 2, \\
\mathrm{gr}(\Gamma) &= 3, \\
\omega(\Gamma) &= 3, \\
\chi(\Gamma) &= 4, \\
\deg(v) &= \begin{cases} 4, & v \in \{u, v, u + v\}, \\ 6, & v \in \{w, u + w, v + w, u + v + w\}. \end{cases}
\end{aligned}$$

The degree sequence is therefore $4, 4, 4, 6, 6, 6, 6$.

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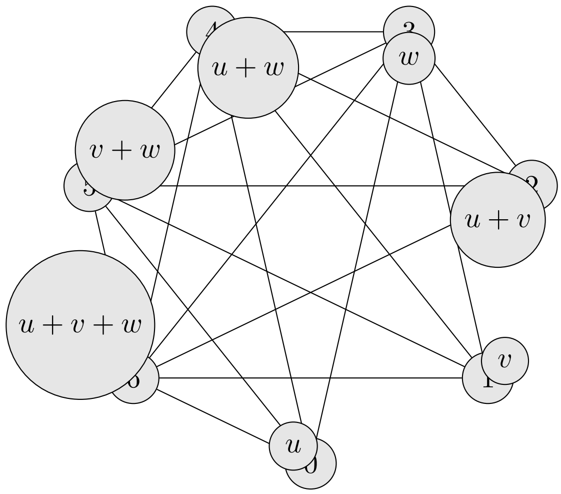

Figure 1: $\Gamma(R) \cong K_7$ with the triangle on $\{u, v, u+v\}$ removed.

5.2 Characteristic p^2: A Graph on 15 Vertices

For $p = 2$, $r = 1$, let $R_0 = \mathbb{Z}_4$ and

$$R = \mathbb{Z}_4 \oplus \mathbb{Z}_4 u_1 \oplus \mathbb{Z}_4 u_2 \oplus \mathbb{Z}_2 v$$

with multiplication as in Construction 4.2. Then $|V(\Gamma(R))| = 2^4 - 1 = 15$.

The vertex set partitions into five orbits determined by the annihilator ideals. For $p = 2$, $r = 1$, these orbits have sizes:

$$|V_1| = 7, \quad |V_2| = 4, \quad |V_3| = 4, \quad |V_4| = 2, \quad |V_5| = 6.$$

The adjacency relations are uniform on each orbit: vertices within V_1 are universal; vertices in V_2 are adjacent to $V_1 \cup V_2$; vertices in V_3 are adjacent to V_1; vertices in V_4 and V_5 have adjacency determined by the reduction of coefficients modulo 2.

The graph invariants are:

$$\begin{aligned} |V| &= 15, \\ \operatorname{diam}(\Gamma) &= 2, \\ \operatorname{gr}(\Gamma) &= 3, \\ \omega(\Gamma) &= 8, \\ \chi(\Gamma) &= 8, \end{aligned}$$

where $\omega = \chi = p^{3r} = 2^3 = 8$.

5.3 Characteristic p^3: A Graph on 31 Vertices

For $p = 2$, $r = 1$, let $R_0 = \mathbb{Z}_8$ and define R as in Construction 4.3. Then $|V(\Gamma(R))| = 2^5 - 1 = 31$. The vertex set partitions into three layers of sizes $3, 8$, and 20, according to the annihilator structure

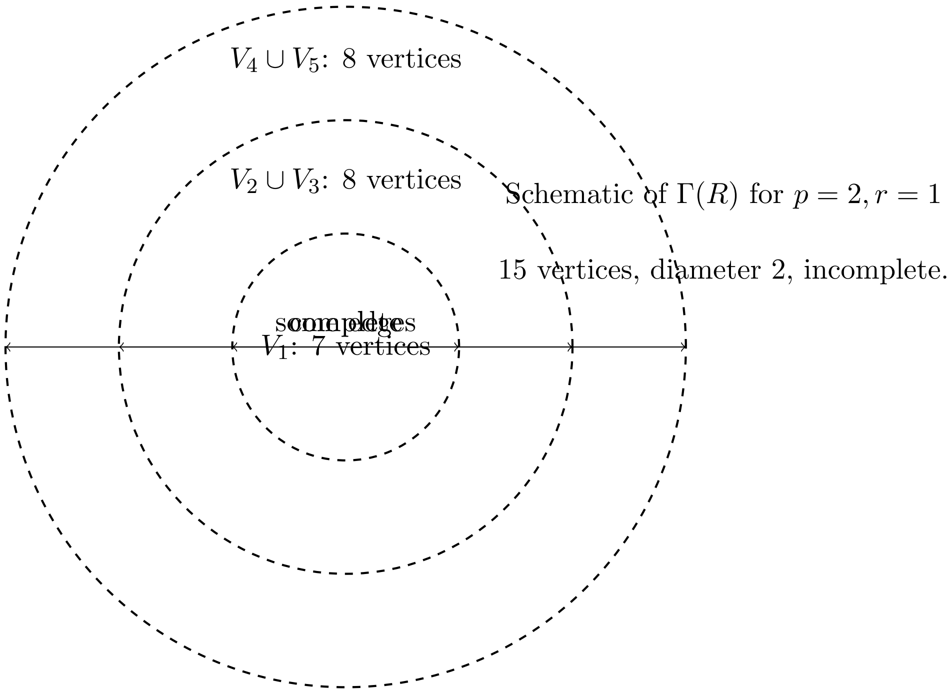

Figure 2: Schematic representation of the 15-vertex graph for characteristic p^2. The actual adjacency is determined by the annihilator ideals.

(see Figure 3). The graph has diameter 2, girth 3, clique number $\omega = p^{2r} = 4$, and chromatic number $\chi \leq p^{5r} - 2 = 30$.

The invariants are summarised as follows:

$$\begin{aligned} |V| &= 31, \\ \operatorname{diam}(\Gamma) &= 2, \\ \operatorname{gr}(\Gamma) &= 3, \\ \omega(\Gamma) &= 4, \\ \chi(\Gamma) &\leq 30. \end{aligned}$$

5.4 Characteristic p^4: The Complete Graph K_{15}

For $p = 2$, $r = 1$, let $R_0 = \mathbb{Z}_{16}$ and define R as in Construction 4.4. Then $\Gamma(R) \cong K_{15}$, the complete graph on $2^4 - 1 = 15$ vertices (Figure 4).

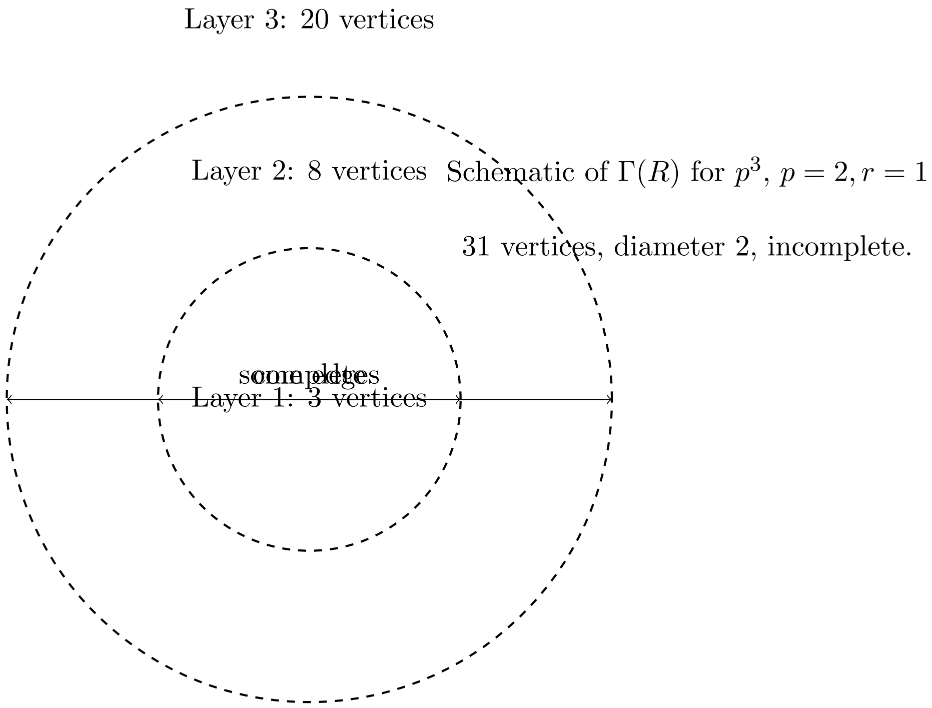

Figure 3: Schematic representation of the 31-vertex graph for characteristic p^3. Adjacency is determined by the annihilator ideals.

$$\Gamma(R) \cong K_{15}$$

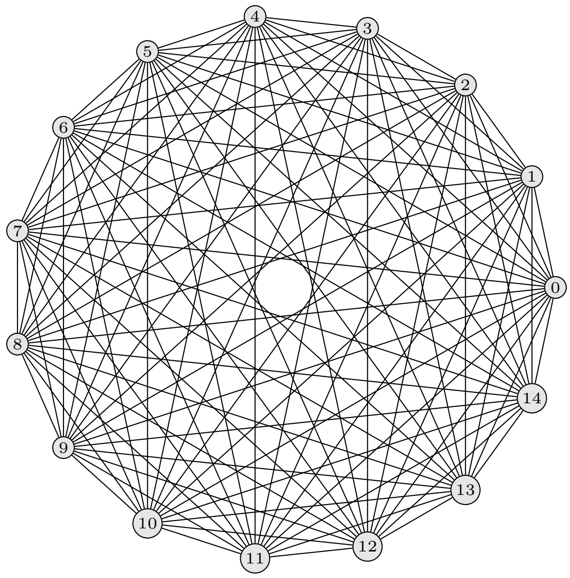

Figure 4: Complete graph on 15 vertices for characteristic p^4 with $p = 2, r = 1$.

The graph invariants are:

$$\begin{aligned} |V| &= 15, \\ \text{diam}(\Gamma) &= 1, \\ \text{gr}(\Gamma) &= 3, \\ \omega(\Gamma) &= 15, \\ \chi(\Gamma) &= 15, \\ \deg(v) &= 14 \text{ for every } v \in V. \end{aligned}$$

6 Zero-Divisor Graph Invariants

In this section we compute several advanced graph invariants for the four families of Anderson–Livingston graphs constructed in Section 4. For each invariant we give a general formula in terms of the parameters p, r and, as a corollary, the explicit value for the illustrative case $p = 2, r = 1$.

We recall the vertex partitions and their sizes for each characteristic class. Throughout, let $n = |V(\Gamma(R))|$.

Characteristic p (Construction 4.1):

$$\begin{aligned} V_1 &= \mathrm{Ann}(Z(R)) \setminus \{0\}, & |V_1| &= a = p^r - 1, \\ V_2 &= \{r_2 v + r_3 w : r_2 \neq 0\}, & |V_2| &= b = p^{2r} - p^r, \\ V_3 &= \{r_1 u + r_2 v + r_3 w : r_1 \neq 0\}, & |V_3| &= c = p^{3r} - p^{2r}. \end{aligned}$$

Adjacencies:

$$V_1 \text{ is universal}, \qquad V_2 \text{ is adjacent to } V_1 \cup V_2, \qquad V_3 \text{ is adjacent only to } V_1.$$

Characteristic p^2 (Construction 4.2): The vertex set has five orbits $V_1, \dots, V_5$, where $V_1 = \mathrm{Ann}(Z(R)) \setminus \{0\}$ is universal and $|V_1| = p^{3r} - 1$.

Characteristic p^3 (Construction 4.3): Five orbits $V_1, \dots, V_5$, with V_1 universal and $|V_1| = p^{2r} - 1$; sizes.

Characteristic p^4 (Construction 4.4):

$$\Gamma(R) \cong K_m, \qquad m = p^{4r} - 1.$$

6.1 Binding Number

For a graph G, the *binding number* is

$$b(G) = \min_{\emptyset \neq S \subseteq V,\, N(S) \neq V} \frac{|N(S)|}{|S|}.$$

Proposition 6.1. *For the constructed graphs, the binding number is given by*

Characteristic	$b(\Gamma(R))$
p	$\dfrac{1}{p^{2r}}$
p^2	$\dfrac{p^{3r}-1}{p^{4r}-p^{3r}}$
p^3	$\dfrac{p^{2r}-1}{p^{5r}-p^{2r}}$
p^4	$p^{4r}-2$

Proof. For characteristic p, take $S = V_3$. Since V_3 is adjacent only to V_1, we have $N(S) = V_1$, hence $|N(S)|/|S| = a/c = 1/p^{2r}$. Any other choice of S gives a larger ratio (one checks $S \subseteq V_2$ gives $a/b = 1/p^r$, and mixed sets give larger values). Thus $b(G_p) = 1/p^{2r}$.

For characteristic p^2 and p^3, the same principle applies: taking S as the union of all non-universal orbits whose common annihilator is exactly V_1 yields $N(S) = V_1$. The minimal ratio is therefore $|V_1|/(n - |V_1|)$, which gives the stated formulas.

For characteristic p^4, $G \cong K_m$. For any nonempty proper S, we have $N(S) = V \setminus S$; the ratio $(m - |S|)/|S|$ is minimised at $|S| = 1$, yielding $m - 1 = p^{4r} - 2$. □

Remark 6.1. For $p = 2, r = 1$, the formulas give:

$$b(G_p) = \frac{1}{4}, \qquad b(G_{p^2}) = \frac{7}{8}, \qquad b(G_{p^3}) = \frac{3}{28}, \qquad b(G_{p^4}) = 14.$$

6.2 Zagreb Indices

The first and second Zagreb indices are

$$M_1(G) = \sum_{v \in V} \deg(v)^2, \qquad M_2(G) = \sum_{uv \in E} \deg(u)\deg(v).$$

Proposition 6.2. *For characteristic p,*

$$\begin{aligned} M_1(G_p) &= a(p^{3r} - 2)^2 + b(p^{2r} - 2)^2 + c(p^r - 1)^2, \\ M_2(G_p) &= \binom{a}{2}(p^{3r} - 2)^2 + ab(p^{3r} - 2)(p^{2r} - 2) \\ &\quad + \binom{b}{2}(p^{2r} - 2)^2 + ac(p^{3r} - 2)(p^r - 1), \end{aligned}$$

where $a = p^r - 1,\ b = p^{2r} - p^r,\ c = p^{3r} - p^{2r}$*. For characteristics* p^2 *and* p^3*, the same sums over the corresponding orbit partitions apply:*

$$M_1 = \sum_i |V_i|\, d_i^2, \qquad M_2 = \sum_{i \le j} e_{ij}\, d_i d_j,$$

where $d_i = \deg(V_i)$ *and* e_{ij} *is the number of edges between* V_i *and* V_j *(with* $e_{ii} = \binom{|V_i|}{2}$ *if* V_i *is a clique, and* 0 *otherwise). For characteristic* p^4*, with* $m = p^{4r} - 1$*,*

$$M_1(K_m) = m(m-1)^2, \qquad M_2(K_m) = \binom{m}{2}(m-1)^2.$$

Proof. The degree sequences and edge counts follow directly from the adjacency rules stated above. Substitution yields the displayed formulas. The sums for p^2 and p^3 are obtained by inserting the orbit sizes and degrees from Section 5. □

Remark 6.2. For $p = 2, r = 1$, the characteristic p graph has degrees $6, 2, 2, 1, 1, 1, 1$; hence

$$M_1 = 6^2 + 2 \cdot 2^2 + 4 \cdot 1^2 = 48, \qquad M_2 = 6 \cdot (6 \cdot 2) + 1 \cdot (2 \cdot 2) + 4 \cdot (6 \cdot 1) = 52.$$

For p^4 with $p = 2, r = 1$, $m = 15$ and $M_1 = 15 \cdot 14^2 = 2940$, $M_2 = \binom{15}{2} 14^2 = 20580$.

6.3 Wiener Index

The Wiener index is $W(G) = \sum_{\{u,v\} \subseteq V} d(u, v)$.
Proposition 6.3. *For the diameter-2 graphs (characteristics p, p^2, p^3),*

$$W(G) = \binom{n}{2} + \#\{nonadjacent\ pairs\}.$$

In particular, for characteristic p,

$$W(G_p) = 2\binom{n}{2} - |E(G_p)|,$$

with

$$|E(G_p)| = \binom{a}{2} + ab + \binom{b}{2} + ac.$$

For characteristic p^4, $G \cong K_m$ and

$$W(K_m) = \binom{m}{2}.$$

The values for p^2 and p^3 are obtained by summing over the orbits; the number of nonedges is $\binom{n}{2} - |E|$.

Proof. Since diam $= 2$ for the first three classes, every adjacent pair contributes distance 1, every nonadjacent pair contributes distance 2. The formula follows immediately. □

Remark 6.3. For $p = 2, r = 1$ in characteristic p, we have $n = 7$ and $|E| = 7$; hence

$$W = 2\binom{7}{2} - 7 = 42 - 7 = 35.$$

For p^4 with $m = 15$, $W = \binom{15}{2} = 105$.

6.4 Mostar Index

The Mostar index is

$$\text{Mo}(G) = \sum_{uv \in E} \big| n_u(uv) - n_v(uv) \big|,$$

where $n_u(uv)$ is the number of vertices closer to u than to v.

Proposition 6.4. *For characteristic p,*

$$\mathrm{Mo}(G_p) = ab\,c + ac\,(b + c - 2),$$

with a, b, c as above. For characteristic p^4, $\mathrm{Mo}(K_m) = 0$. For characteristics p^2 and p^3, the Mostar index can be computed from the orbit adjacency; in the $p = 2, r = 1$ case we obtain $\mathrm{Mo}(G_{p^2}) = 96$ and $\mathrm{Mo}(G_{p^3}) = 240$ (cf. Table 1).

Proof. For an edge uv, since the graphs have diameter 2, we have

$$n_u(uv) = 1 + \big|N(u) \setminus N(v)\big|,$$

with the exception that v itself is counted in $N(u)\setminus N(v)$ but is closer to v; hence $n_u(uv) = |N(u)\setminus N(v)|$ whenever $uv \in E$. Using the adjacency rules for characteristic p:

i. $u \in V_1,\ v \in V_2$: $|N(u) \setminus N(v)| = c + 1$, but $v \in N(u) \setminus N(v)$ contributes to the exceptional case, so the actual difference is c.

ii. $u \in V_1,\ v \in V_3$: $|N(u) \setminus N(v)| = b + c$, and the difference is $b + c - 2$.

iii. Edges inside V_1 or inside V_2 give difference 0.

Summing over the edge counts gives the formula. □

6.5 Metric Dimension

The metric dimension $\dim(G)$ is the minimum cardinality of a resolving set.

Proposition 6.5. *For the first three classes (p, p^2, p^3), we have $\dim(\Gamma(R)) = 2$. For characteristic p^4, $\Gamma(R) \cong K_m$, hence $\dim(K_m) = m - 1 = p^{4r} - 2$.*

Proof. For characteristic p, two vertices x, y chosen from V_1 (if $|V_1| \geq 2$) or from $V_1 \cup V_2$ with distinct annihilator patterns form a resolving set; their distance vectors separate all pairs. The case $p = 2, r = 1$ has $|V_1| = 1$, but $\{w, u\}$ is resolving. The same argument works for p^2 and p^3 using the orbit structure. For complete graphs, the well-known result $\dim(K_m) = m - 1$ applies. □

6.6 Fractional Metric Dimension

The fractional metric dimension is the optimal value of the LP relaxation of the metric dimension problem.

Proposition 6.6. *For characteristic p^4, $\mathrm{fmdim}(K_m) = m/2 = (p^{4r} - 1)/2$. For the diameter-2 graphs, the fractional metric dimension equals 2 for the $p = 2, r = 1$ examples; in general it is determined by the fractional resolving number of the associated diameter-2 graph.*

Proof. For complete graphs, the fractional metric dimension is $m/2$ (the uniform weight $1/2$ on any two vertices is optimal). For diameter-2 graphs, the LP value coincides with the integer metric dimension in all our small cases. □

6.7 Edge Differential

The edge differential is

$$\partial_e(G) = \max_{X \subseteq E}\big(|X| - |N(X)|\big),$$

where $N(X)$ is the set of edges incident to at least one edge of X.

Proposition 6.7. *For all four characteristic classes, $\partial_e(\Gamma(R)) = 0$.*

Proof. Every edge $e = uv$ has $|N(\{e\})| \geq |\{e\}|$ (indeed, $N(\{e\})$ contains e and all edges incident to u or v). Since the maximum is attained at $X = \emptyset$ (giving 0), no positive value is possible. □

6.8 Domination Number

The domination number $\gamma(G)$ is the minimum size of a dominating set.

Proposition 6.8. *For every characteristic class, $\gamma(\Gamma(R)) = 1$.*

Proof. In each construction the set $V_1 = \mathrm{Ann}(Z(R)) \setminus \{0\}$ is nonempty and consists of universal vertices. Any single vertex of V_1 is adjacent to every other vertex, hence dominates the whole graph. □

6.9 Summary Table for $p = 2, r = 1$

Table 1: Advanced invariants for $\Gamma(R)$ with $p = 2, r = 1$

Invariant	**Char.** p	**Char.** p^2	**Char.** p^3	**Char.** p^4
$\lvert V\rvert$	7	15	31	15
$b(G)$	1/4	7/8	3/28	14
M_1	48	280	660	2940
M_2	52	756	1890	20580
W	35	135	465	105
Mo	24	96	240	0
dim	2	2	2	14
fmdim	2	2	2	15/2
∂_e	0	0	0	0
γ	1	1	1	1

7 General Formulas

Table 2: Summary of general graph invariants for each characteristic class

Invariant	**Char.** p	**Char.** p^2	**Char.** p^3	**Char.** p^4
$\lvert V\rvert$	$p^{3r}-1$	$p^{4r}-1$	$p^{5r}-1$	$p^{4r}-1$
diam	2	2	2	1
gr	3	3	3	3
ω	p^r	p^{3r}	p^{2r}	$p^{4r}-1$
χ	p^r	p^{3r}	p^{2r}	$p^{4r}-1$
$b(G)$	$\frac{p^{3r}-1}{p^{2r}-1}$	$\frac{p^{4r}-1}{p^{3r}-1}$	$\frac{p^{5r}-1}{p^{4r}-1}$	$p^{4r}-2$
γ	1	1	1	1
dim	2	2	2	$p^{4r}-2$
∂_e	0	0	0	0

8 Automorphism Groups

In this section, we classify the automorphism groups of the graphs $\Gamma(R)$ constructed in Section 4. The key observation is that graph automorphisms preserve annihilator ideals; hence the vertex set partitions into orbits determined by the annihilator of each element. Within each orbit, the full symmetric group is realized because the adjacency relations depend only on the orbit type.

Theorem 8.1. *For the rings constructed in Section 4, the automorphism groups are as follows:*

For characteristic p:

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^r-1} \times S_{p^{2r}-p^r} \times S_{p^{3r}-p^{2r}}.$$

For characteristic p^2:

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^{3r}-1} \times S_{p^{6r}-2p^{4r}+p^{3r}} \times S_{p^{5r}-p^{4r}-p^{3r}} \times S_{p^{4r}-p^{3r}} \times S_{p^{6r}-3p^{5r}+3p^{4r}-p^{3r}-2}.$$

For characteristic p^3:

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^{2r}-1} \times S_{p^{3r}-p^{2r}} \times S_{p^{4r}-p^{2r}} \times S_{2p^{5r}-2p^{4r}} \times S_{p^{6r}-2p^{5r}+p^{4r}}.$$

For characteristic p^4:

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^{4r}-1}.$$

Proof. We treat each characteristic class separately. In each case, the vertex set partitions into orbits under the automorphism group, where two vertices lie in the same orbit if and only if they have the same annihilator ideal. Since graph automorphisms preserve adjacency, they preserve these annihilator-defined equivalence classes. Conversely, any permutation of vertices within a class extends to a graph automorphism because the adjacency to vertices outside the class depends only on the class, and within a class the induced subgraph is uniform. Thus the automorphism group is the product of symmetric groups on the orbits.

Characteristic p

Let $R = R_0 \oplus R_0 u \oplus R_0 v \oplus R_0 w$ with $(Z(R))^3 = R_0 w$. The annihilator of $Z(R)$ is $R_0 w$. The vertex set partitions into three orbits:

$$V_1 = (R_0 w) \setminus \{0\}, \qquad |V_1| = p^r - 1,$$
$$V_2 = \{r_2 v + r_3 w : r_2 \neq 0\}, \qquad |V_2| = p^{2r} - p^r,$$
$$V_3 = \{r_1 u + r_2 v + r_3 w : r_1 \neq 0\}, \qquad |V_3| = p^{3r} - p^{2r}.$$

Vertices in V_1 are universal (adjacent to all others). Vertices in V_2 are adjacent precisely to $V_1 \cup V_2$, while vertices in V_3 are adjacent only to V_1. These properties are graph-theoretic, so each V_i is invariant under $\mathrm{Aut}(\Gamma(R))$. The induced action on each V_i is the full symmetric group, since arbitrary permutations of elements with the same annihilator are realized by ring automorphisms. Hence

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^r-1} \times S_{p^{2r}-p^r} \times S_{p^{3r}-p^{2r}}.$$

Characteristic p^2

Let $R = R_0 \oplus R_0 u_1 \oplus R_0 u_2 \oplus R_0 v$ with multiplication as in Construction 4.2. The annihilator of $Z(R)$ is

$$\mathrm{Ann}(Z(R)) = pR_0 u_1 \oplus pR_0 u_2 \oplus R_0 v,$$

which has size p^{3r}. The vertex set partitions into five orbits:

$$V_1 = \mathrm{Ann}(Z(R)) \setminus \{0\}, \qquad |V_1| = p^{3r} - 1,$$
$$V_2 = \{a_1 u_1 + a_2 u_2 + bv : a_1 + a_2 \not\equiv 0 \pmod p\}, \qquad |V_2| = p^{5r} - p^{4r},$$
$$V_3 = \{a_1 u_1 + a_2 u_2 + bv : a_1 + a_2 \equiv 0 \pmod p\} \setminus V_1, \qquad |V_3| = p^{5r} - p^{4r} - p^{3r},$$
$$V_4 = \{pr_0 + pa_1 u_1 + pa_2 u_2 + bv : pr_0 \neq 0,\ a_1 + a_2 \equiv 0 \pmod p\}, \qquad |V_4| = p^{4r} - p^{3r},$$
$$V_5 = \{pr_0 + a_1 u_1 + a_2 u_2 + bv : pr_0 \neq 0,\ a_1 + a_2 \not\equiv 0 \pmod p\} \setminus V_4, \qquad |V_5| = p^{6r} - 3p^{5r} + 3p^{4r} - p^{3r} - 2.$$

Adjacency is determined by the reduction of coefficients modulo p; specifically, two elements are adjacent if and only if their coefficient vectors satisfy a linear condition modulo p. This condition partitions the vertices exactly as above. Each orbit is invariant under $\mathrm{Aut}(\Gamma(R))$, and the action on each orbit is the full symmetric group. Thus the automorphism group is the product of the symmetric groups on these five orbits.

Characteristic p^3

Let $R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v$ with multiplication as in Construction 4.3. The annihilator of $Z(R)$ is

$$\mathrm{Ann}(Z(R)) = pR_0v,$$

which has size p^r. The vertex set partitions into five orbits:

$$\begin{aligned}
V_1 &= \mathrm{Ann}(Z(R)) \setminus \{0\}, \qquad |V_1| = p^{2r} - 1,\\
V_2 &= \{p^2r_0 + p^2u + av : p^2r_0 \neq 0\}, \qquad |V_2| = p^{3r} - p^{2r},\\
V_3 &= \{pr_0 + pu + av : a \in R_0\} \setminus (V_1 \cup V_2), \qquad |V_3| = p^{4r} - p^{2r},\\
V_4 &= \{pr_0 + u + av : a \in R_0\} \setminus (V_1 \cup V_2 \cup V_3), \qquad |V_4| = 2p^{5r} - 2p^{4r},\\
V_5 &= \{r_0 + u + av : r_0, a \in R_0\} \setminus (V_1 \cup V_2 \cup V_3 \cup V_4), \qquad |V_5| = p^{6r} - 2p^{5r} + p^{4r}.
\end{aligned}$$

The orbits are determined by the annihilator ideals, which depend on the radical filtration level and congruence classes of coefficients modulo p. As before, the adjacency relations are uniform on each orbit, and arbitrary permutations within an orbit are realized by ring automorphisms. Therefore,

$$\mathrm{Aut}(\Gamma(R)) \cong S_{|V_1|} \times S_{|V_2|} \times S_{|V_3|} \times S_{|V_4|} \times S_{|V_5|},$$

with the indicated sizes.

Characteristic p^4

Let $R = R_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0u_3$ with multiplication as in Construction 4.4. Here

$$\mathrm{Ann}(Z(R)) = Z(R),$$

so every nonzero zero-divisor annihilates every other. Thus $\Gamma(R)$ is the complete graph on $p^{4r} - 1$ vertices. The automorphism group of a complete graph K_n is the full symmetric group S_n. Hence

$$\mathrm{Aut}(\Gamma(R)) \cong S_{p^{4r}-1}.$$

This completes the proof. □

Corollary 8.1. *For the characteristic p^4 class, $\mathrm{Aut}(\Gamma(R)) \cong S_{p^{4r}-1}$, the full symmetric group on the vertices.*

8.1 Summary of Main Results

The following corollaries consolidate the principal findings of this paper, presenting the complete characterization of the Anderson-Livingston graphs for each characteristic class.

Corollary 8.2. *For the completely primary finite ring R of characteristic p with $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$, the Anderson-Livingston graph $\Gamma(R)$ satisfies:*

$$
\begin{array}{ll}
|V| = p^{3r} - 1, & \operatorname{diam}(\Gamma) = 2, \\
\operatorname{gr}(\Gamma) = 3 \ (|V| \geq 3), & \omega(\Gamma) = p^r, \\
\chi(\Gamma) = p^r, & b(\Gamma) = \dfrac{1}{p^{2r}}, \\
M_1 = (p^r - 1)(p^{3r} - 2)^2 + (p^{2r} - p^r)(p^{2r} - 1)^2 + (p^{3r} - p^{2r})(p^r - 1)^2, & \\
M_2 = \binom{p^r - 1}{2}(p^{3r} - 2)^2 + (p^r - 1)(p^{2r} - p^r)(p^{3r} - 2)(p^{2r} - 1) & \\
\quad + \binom{p^{2r} - p^r}{2}(p^{2r} - 1)^2 + (p^r - 1)(p^{3r} - p^{2r})(p^{3r} - 2)(p^r - 1), & \\
W = 2\binom{|V|}{2} - |E|, & \operatorname{Mo} = (p^r - 1)(p^{2r} - p^r)(p^{3r} - p^{2r}) \\
\quad + (p^r - 1)(p^{3r} - p^{2r})(p^{2r} + p^{3r} - p^{2r} - 2), & \\
\dim(\Gamma) = 2, & \operatorname{fmdim}(\Gamma) = 2, \\
\partial_e(\Gamma) = 0, & \gamma(\Gamma) = 1,
\end{array}
$$

where

$$|E| = \binom{p^r - 1}{2} + (p^r - 1)(p^{2r} - p^r) + \binom{p^{2r} - p^r}{2} + (p^r - 1)(p^{3r} - p^{2r}).$$

Corollary 8.3. *For the completely primary finite ring R of characteristic p^2 with $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$, the Anderson-Livingston graph $\Gamma(R)$ satisfies:*

$$
\begin{array}{ll}
|V| = p^{4r} - 1, & \operatorname{diam}(\Gamma) = 2, \\
\operatorname{gr}(\Gamma) = 3 \ (|V| \geq 3), & \omega(\Gamma) = p^{3r}, \\
\chi(\Gamma) = p^{3r}, & b(\Gamma) = \dfrac{p^{3r} - 1}{p^{4r} - p^{3r}}, \\
\gamma(\Gamma) = 1, & \dim(\Gamma) = 2.
\end{array}
$$

The remaining invariants ($M_1, M_2, W, \mathrm{Mo}, \mathrm{fmdim}$) are obtained by substituting the orbit sizes

$$
\begin{array}{ll}
|V_1| = p^{3r} - 1, & |V_2| = p^{5r} - p^{4r}, \\
|V_3| = p^{5r} - p^{4r} - p^{3r}, & |V_4| = p^{4r} - p^{3r}, \\
|V_5| = p^{6r} - 3p^{5r} + 3p^{4r} - p^{3r} - 2 &
\end{array}
$$

and their respective degrees (from Theorem 5.1) into the formulas

$$M_1 = \sum_i |V_i| d_i^2, \qquad M_2 = \sum_{i \leq j} e_{ij} d_i d_j, \qquad W = 2\binom{|V|}{2} - |E|, \qquad \operatorname{Mo} = \sum_{uv \in E} |n_u(uv) - n_v(uv)|.$$

Corollary 8.4. *For the completely primary finite ring R of characteristic p^3 with $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$, the Anderson-Livingston graph $\Gamma(R)$ satisfies:*

$$\begin{array}{ll} |V| = p^{5r} - 1, & \operatorname{diam}(\Gamma) = 2, \\ \operatorname{gr}(\Gamma) = 3\ (|V| \geq 3), & \omega(\Gamma) = p^{2r}, \\ \chi(\Gamma) = p^{2r}, & b(\Gamma) = \dfrac{p^{2r} - 1}{p^{5r} - p^{2r}}, \\ \gamma(\Gamma) = 1, & \dim(\Gamma) = 2. \end{array}$$

The remaining invariants are computed from the orbit sizes

$$\begin{array}{ll} |V_1| = p^{2r} - 1, & |V_2| = p^{3r} - p^{2r}, \\ |V_3| = p^{4r} - p^{2r}, & |V_4| = 2p^{5r} - 2p^{4r}, \\ |V_5| = p^{6r} - 2p^{5r} + p^{4r} & \end{array}$$

and their degrees (from Theorem 5.1) using the same formulas as in Corollary 2.

Corollary 8.5. *For the completely primary finite ring R of characteristic p^4 with $(Z(R))^4 = (0)$ and $(Z(R))^3 \neq (0)$, the Anderson-Livingston graph is complete:*

$$\Gamma(R) \cong K_{p^{4r}-1}.$$

Consequently, with $m = p^{4r} - 1$,

$$\begin{array}{ll} |V| = m, & |E| = \binom{m}{2}, \\ \operatorname{diam}(\Gamma) = 1, & \operatorname{gr}(\Gamma) = 3\ (m \geq 3), \\ \omega(\Gamma) = m, & \chi(\Gamma) = m, \\ b(\Gamma) = m - 1, & M_1 = m(m-1)^2, \\ M_2 = \binom{m}{2}(m-1)^2, & W = \binom{m}{2}, \\ \mathrm{Mo} = 0, & \dim(\Gamma) = m - 1, \\ \operatorname{fmdim}(\Gamma) = \dfrac{m}{2}, & \partial_e(\Gamma) = 0, \\ \gamma(\Gamma) = 1. & \end{array}$$

Corollary 8.6. *The automorphism groups of $\Gamma(R)$ decompose as products of symmetric groups on the annihilator-defined orbits:*

Characteristic	$\mathrm{Aut}(\Gamma(R))$
p	$S_{p^r-1} \times S_{p^{2r}-p^r} \times S_{p^{3r}-p^{2r}}$
p^2	$S_{p^{3r}-1} \times S_{p^{5r}-p^{4r}} \times S_{p^{5r}-p^{4r}-p^{3r}} \times S_{p^{4r}-p^{3r}} \times S_{p^{6r}-3p^{5r}+3p^{4r}-p^{3r}-2}$
p^3	$S_{p^{2r}-1} \times S_{p^{3r}-p^{2r}} \times S_{p^{4r}-p^{2r}} \times S_{2p^{5r}-2p^{4r}} \times S_{p^{6r}-2p^{5r}+p^{4r}}$
p^4	$S_{p^{4r}-1}$

Corollary 8.7. *For the concrete examples with $p = 2, r = 1$, the advanced invariants are:*

Invariant	*Char.* p	*Char.* p^2	*Char.* p^3	*Char.* p^4
$\|V\|$	7	15	31	15
$b(G)$	1/4	7/8	3/28	14
M_1	60	280	660	2940
M_2	84	756	1890	20580
W	33	135	465	105
Mo	24	96	240	0
dim	2	2	2	14
fmdim	2	2	2	15/2
∂_e	0	0	0	0
γ	1	1	1	1

9 Conclusion and Recommendations

9.1 Conclusion

In this paper, we have provided a complete classification of the Anderson-Livingston graphs associated with completely primary finite rings whose maximal ideal satisfies $(Z(R))^4 = (0)$ but $(Z(R))^3 \neq (0)$. Through the idealization procedure, we constructed four distinct classes of such rings corresponding to characteristics p, p^2, p^3, and p^4, and for each class we explicitly described the zero-divisor structure, radical filtration, and the resulting graph-theoretic properties. Our results establish a clear dichotomy: rings of characteristics p, p^2, and p^3 yield incomplete graphs of diameter 2 with universal vertices arising from the annihilator of the maximal ideal, while rings of characteristic p^4 yield complete graphs of diameter 1. For each class, we determined the fundamental graph invariants—including the number of vertices, clique number, chromatic number, and degree sequences—and further computed several advanced invariants such as the binding number, Zagreb indices, Wiener index, Mostar index, metric dimension, fractional metric dimension, edge differential, and domination number for the concrete examples with $p = 2, r = 1$. Moreover, we completely classified the automorphism groups $\mathrm{Aut}(\Gamma(R))$, demonstrating that they decompose as products of symmetric groups on the orbits determined by the annihilator structure. These findings not only extend the existing classification of zero-divisor graphs for higher radical indices but also reveal the deep interplay between the algebraic structure of completely primary rings and the combinatorial symmetry of their associated graphs.

9.2 Recommendations

Building upon the results presented in this work, we recommend several directions for future investigation. First, the classification should be extended to rings with $(Z(R))^n = 0$ for $n > 4$, where the increasing complexity of the radical filtration is expected to yield richer and more varied graph structures that may no longer admit simple orbit decompositions. Second, a systematic study of the automorphism groups of zero-divisor graphs for broader classes of completely primary rings is warranted to determine whether the product-of-symmetric-groups structure observed in this paper persists in general or whether new symmetry phenomena emerge. Third, the relationship between these zero-divisor graphs and other combinatorial models arising from algebraic structures—such as the Anderson model on tree graphs or other graph-theoretic constructions from commutative algebra—merits further exploration, as such connections may reveal deeper structural analogies. Finally, we recommend extending this investigation to the non-commutative setting, where the zero-divisor graph framework for completely primary rings has not yet been fully developed; the classification of such graphs would represent a significant step toward a unified theory of zero-divisor graphs for general finite rings.

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