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A Characterization of Algebraic Structures of Group Rings obtained from Matrix Rings

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Abstract

Let R be a commutative ring with unity and G a finite group. We investigate when a group ring $R[G]$ is isomorphic to a full matrix ring $M_n(S)$ for some (possibly non-commutative) ring S. For commutative R with $|G|$ invertible, we prove that such an isomorphism forces G to be trivial when $n > 1$. Thus the only non-trivial matrix ring isomorphisms for group rings occur when $|G|$ is not invertible in R or when R itself is non-commutative. We also give a complete description of embeddings $R[G] \hookrightarrow M_n(T)$ as direct summands in terms of normal subgroups, and we discuss consequences for Morita equivalence and maximal orders. Examples illustrate the optimality of our conditions, and we conclude with open problems.

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1 Introduction

The theory of group rings has enjoyed a vigorous and fruitful development ever since the seminal work of Cayley, who first recognised that the elements of a group can be considered as formal linear combinations over the integers. The fundamental problem of understanding how structural information about the group G is encoded in the algebraic structure of its group ring $R[G]$ has motivated an extensive research programme, spanning topics from the classical isomorphism problem to the intricate study of the augmentation ideal and the unit group. While much of the classical literature, culminating in the authoritative treatise *The Algebraic Structure of Group Rings* by Passman [6], has concentrated on the ring-theoretic properties induced by the group structure, the converse question—namely, how the matrix ring structure of $R[G]$ reflects the underlying group and coefficient ring—has received comparatively less systematic attention. The present article addresses this gap by providing a comprehensive characterisation of algebraic structures of group rings that are obtained from matrix rings.

The foundational work in the area is, of course, the corpus of Passman [6], which established deep links between the group-theoretic nature of G and the ring-theoretic properties of $R[G]$. Central to this is the role of the augmentation ideal $\Delta(G)$, a two-sided ideal that encodes the group's abelianisation, and its interplay with polynomial identities and the Jacobson radical. Passman also treated, albeit in passing, the conditions under which a group ring can be expressed as a direct product of matrix rings over fields, particularly when R is a field of characteristic zero and G is finite—a setting where Maschke's theorem applies. However, the question of when $R[G]$ itself is *isomorphic* to a full matrix ring $M_n(S)$ for some (potentially non-commutative) ring S was not at the centre of that investigation. The literature has thus largely focused on the decomposition of group algebras as direct sums of matrix rings over division algebras (the classical Artin–Wedderburn picture), but the more elementary question of whether the whole ring can be presented as a single matrix block over an arbitrary base ring has been left largely unexplored. This omission leaves a notable lacuna: a systematic account of the precise conditions under which a group ring can be thought of as a matrix ring over some base is still missing.

A parallel line of inquiry, championed by Sehgal, Milies and their collaborators, has examined the unit groups of group rings and their matrix representations. The monograph *Topics in Group Rings* by Sehgal [8] and the survey by Milies [5] on units of group rings provide a wealth of results on the structure of the unit group $U(R[G])$ and its embedding into matrix groups over R. In particular, Sehgal and Marciniak studied finite matrix groups over nilpotent group rings, showing that the matrix group $\mathrm{SGL}_n(\mathbb{Z}H)$ is torsion free when H is a torsion-free nilpotent group. These investigations have produced a rich theory of *matrix representations* of group rings, but they have not attempted to characterise when the group ring itself, as a ring, is isomorphic to a full matrix ring over some base. Consequently, the link between the existence of an embedding $R[G] \hookrightarrow M_n(T)$ and the existence of an isomorphism $R[G] \cong M_n(S)$ remains under-exploited. The present work fills this gap by establishing a necessary and sufficient condition, in terms of a normal subgroup of index n, for the existence of a direct-summand embedding; we then show that such an embedding forces a very rigid structure on G and R. A third, more recent development stems from the work of Karpilovsky [3] on crossed products and graded rings. The theory of crossed products generalises group rings by allowing a twisting 2-cocycle, and Karpilovsky's monograph provides a comprehensive account of the algebraic structure of such objects, including the centre, the Jacobson radical and the prime ideals. From the perspective of the present problem, the group ring $R[G]$ can be regarded as a special case of a crossed product $R * (G/N)$ when N is a normal subgroup. This viewpoint is exploited in Theorem 5.2, where we show that $R[G]$ naturally embeds as a subring of $M_n(R[N])$ via the regular representation on the coset space G/N, and that this embedding is an isomorphism precisely when the extension splits and the cocycle is trivial. The embedding, however, is not a full matrix ring but rather a proper subring consisting of permutation-like matrices, a fact that seems to have been overlooked in earlier attempts to represent group rings as matrix rings (see, e.g., Hurley's construction, discussed next). Thus, while crossed product theory provides the right language, the subtlety lies in distinguishing an embedding from an isomorphism; the present article clarifies this distinction by giving a complete characterisation of the image of the regular representation.

More recent contributions, notably by Hurley [2] and his collaborators, have attempted to establish isomorphisms between group rings and certain subrings of matrix rings. Hurley proved that the group ring RG of a finite group G of order n is isomorphic to a certain subring of $M_n(R)$, called the ring of *group ring matrices*. Dougherty, Gildea and Korban [1] later extended this construction by subdividing the matrices and imposing technical assumptions. While these constructions are mathematically elegant and have found applications in coding theory, they do not, in general, produce an isomorphism between RG and a *full* matrix ring $M_n(S)$; rather, they embed RG as a proper subring of $M_n(R)$ when R is commutative. The question of when the Hurley embedding is surjective—i.e., when RG itself is a full matrix ring—is not addressed in that literature. The present article resolves this question in the negative for commutative R with $|G|$ invertible, proving that RG is *never* a full matrix ring of size $n > 1$ over any ring. Moreover, we treat the non-commutative coefficient ring separately, providing examples (e.g., $R = M_2(\mathbb{F}_2)$ and $G = C_2$) where the group ring does become a full matrix ring.

In light of the above, the present article makes the following contributions. First, we give a short, self-contained proof that for a commutative ring R with $|G|$ invertible, the group ring $R[G]$ with $|G| > 1$ cannot be isomorphic to $M_n(S)$ for any $n > 1$ (Theorem 4.1). This result highlights a fundamental obstruction: the augmentation ideal, when pulled back, yields a commutative quotient R, whereas a matrix ring of size > 1 always has a non-commutative quotient modulo any proper two-sided ideal. Second, we establish that $R[G]$ can be embedded as a direct summand of a matrix ring if and only if G admits a normal subgroup N of index n such that $|N|$ is invertible in R (Theorem 5.2). Third, we provide additional strong results: a characterisation of when $R[G]$ is prime or semiprime in terms of the existence of a matrix representation (Theorem 4.4), a description of the centre of $R[G]$ when it is a matrix ring (Proposition 4.5), and a condition on the Jacobson radical (Proposition 4.6). Finally, we relate our results to Morita theory and to the theory of maximal orders, showing that every finite group algebra over a field of characteristic zero is Morita equivalent to a commutative ring, and that the question of when $R[G]$ is a matrix ring is essentially a question about the existence of a full set of matrix units in $R[G]$. A series of open problems concludes the article, pointing towards further research on the cohomological criterion for such isomorphisms and on the infinite-group case.

2 Literature Review

The study of group rings and their relationship with matrix rings has a long history, and several distinct lines of research have contributed to our current understanding. In this section we critically survey the most relevant works, identify their strengths and weaknesses, and highlight the gaps that the present article addresses. For clarity, we structure the discussion around four thematic areas: classical structure theory, unit group methods, crossed product theory, and recent matrix embedding constructions.

The classical approach, epitomised by the monographs of Passman [6] and of Sehgal [8], has been to understand the ring-theoretic properties of $R[G]$ (primeness, semisimplicity, the Jacobson radical, etc.) in terms of group-theoretic data. Passman's treatment of the augmentation ideal and its powers, the theory of G-graded rings, and the characterisation of when $R[G]$ is prime or semiprime are masterful. However, this body of work is primarily concerned with internal structure: given G and R, what can we say about $R[G]$? The converse problem—given that $R[G]$ happens to be isomorphic to a matrix ring $M_n(S)$, what does that imply about G and S—receives almost no attention. Passman does discuss when $R[G]$ is a direct product of matrix rings over fields (the Artin–Wedderburn decomposition for group algebras), but the possibility of the whole ring being a single matrix block is not investigated. Consequently, the classical literature leaves unanswered the question of whether non-trivial group rings can ever be full matrix rings. Our Theorem 4.1 provides a definitive negative answer for the case where $|G|$ is invertible in the commutative coefficient ring R, thereby filling this gap. Moreover, we show that the obstruction is elementary and stems from the existence of the augmentation ideal.

A second line of research, developed by Sehgal, Marciniak, Milies and others, focuses on the unit group $U(R[G])$ and its matrix representations. The survey by Milies [5] summarises many results on the structure of units in group rings, including the construction of free subgroups, the description of the centre of the unit group, and the embedding of $U(R[G])$ into matrix groups over R. Sehgal and Marciniak showed that for a torsion-free nilpotent group H, the special linear group $\mathrm{SGL}_n(\mathbb{Z}H)$ is torsion free. These results are powerful, but they are about *subgroups* of matrix groups, not about the whole group ring being a matrix ring. In particular, they do not address the question of when the ring $R[G]$ itself admits a full matrix ring structure. The present article shifts the focus from the unit group to the ring as a whole. Our embedding theorem (Theorem 5.2) can be seen as a complementary construction: while the unit group lives inside some $\mathrm{GL}_m(T)$, the entire group ring sits inside a matrix ring over a smaller group ring. This is a different kind of matrix realisation, and we characterise exactly when such an embedding is possible.

The theory of crossed products, as developed by Karpilovsky [3] and others, provides a third perspective. A crossed product $R * G$ is a generalisation of a group ring where multiplication involves a twisting 2-cocycle. For a normal subgroup $N \triangleleft G$, the group ring $R[G]$ can be decomposed as a crossed product $R[N] * (G/N)$. This decomposition is at the heart of the regular representation embedding we use. Karpilovsky's comprehensive treatment covers the centre, the prime ideals, and the Brauer group of crossed products. However, the emphasis is again on describing the structure of the crossed product in terms of the data $(R[N], G/N, \sigma, \tau)$, not on determining when the resulting ring is a full matrix ring. Our contribution is to show that the natural embedding of $R[G]$ into $M_n(R[N])$ obtained from the regular representation is a direct summand if and only if a certain splitting condition holds, and that this embedding is surjective only in trivial cases (when $N = \{1\}$ or when $R[N]$ itself is a matrix ring). Thus we turn the machinery of crossed products into a characterisation tool for matrix ring isomorphisms. Finally, recent work by Hurley [2] and by Dougherty, Gildea and Korban [1] has attempted to construct explicit isomorphisms between group rings and subrings of matrix rings. Hurley showed that for a finite group G of order n, the group ring RG (over a commutative ring R) is isomorphic to a subring of $M_n(R)$ consisting of matrices that are linear combinations of permutation matrices corresponding to the regular representation of G. This is a beautiful embedding, but it is not an isomorphism onto a full matrix ring; the image is a proper subring of $M_n(R)$ unless $R = 0$ or G is trivial. Dougherty et al. extended this to "subdivided" matrices, but still the target is a proper subring. In all these constructions, the question of when the Hurley embedding is surjective (i.e., when $RG \cong M_n(R)$) is not answered. Our Theorem 4.1 shows that for commutative R with $|G|$ invertible, this never happens when $n > 1$. Thus the Hurley embedding is never an isomorphism to a full matrix ring under those hypotheses. We also provide examples where the coefficient ring is non-commutative (e.g., $R = M_2(\mathbb{F}_2)$) and the Hurley embedding becomes an isomorphism onto a full matrix ring over a different base (namely $\mathbb{F}_2[\epsilon]$). This clarifies the precise role of commutativity and invertibility of $|G|$

in such constructions.

In summary, the literature has thoroughly investigated the internal structure of group rings and their unit groups, as well as their decomposition as crossed products, but the specific question of when a group ring can be isomorphic to a full matrix ring over an arbitrary ring has remained largely open. The existing embedding constructions do not address surjectivity, and the classical theorems do not exclude the possibility of non-trivial matrix ring isomorphisms for group rings. The present article fills this void by providing a clean algebraic obstruction (the augmentation ideal) that rules out such isomorphisms for commutative coefficients with invertible group order, and by giving a complete characterisation of embeddings as direct summands. Moreover, we show that when the coefficient ring is allowed to be non-commutative, matrix ring isomorphisms become possible, as illustrated by explicit examples.

3 Preliminaries

3.1 Matrix units and Peirce decomposition

Definition 3.1. A set $\{e_{ij} \mid 1 \leq i, j \leq n\} \subseteq A$ is called a **system of $n \times n$ matrix units** if

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \qquad \text{and} \qquad \sum_{i=1}^{n} e_{ii} = 1_A.$$

Proposition 3.2. *A ring A is isomorphic to $M_n(S)$ for some ring S if and only if it contains a system of $n \times n$ matrix units $\{e_{ij}\}$. In that case, $S \cong e_{11}Ae_{11}$ and the isomorphism is given by $a \mapsto (e_{1i}ae_{j1})_{i,j}$.*

Proof. See [4, §3.2]. □

3.2 Group rings

Let R be a commutative ring and G a finite group. The group ring $R[G]$ consists of formal sums $\sum_{g \in G} a_g g$ with $a_g \in R$. Multiplication is induced by the group law. The **augmentation ideal** is

$$\Delta(G) = \Big\{ \sum_{g \in G} a_g g \in R[G] \;\Big|\; \sum_{g \in G} a_g = 0 \Big\}.$$

For any subgroup $H \leq G$ with $|H|$ invertible in R, the element $\widehat{H} = \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent. If H is normal, then $\widehat{H}$ is central.

When $|G|$ is invertible in R, the centre is

$$Z(R[G]) = \bigoplus_{C\in \mathrm{Cl}(G)} R\cdot e_C, \qquad e_C = \frac{1}{|C|}\sum_{g\in C} g,$$

where $\mathrm{Cl}(G)$ denotes the conjugacy classes of G.

4 When is a group ring a full matrix ring?

We begin with the main negative result.
Theorem 4.1 (A). *Let R be a commutative ring and G a finite group with $|G| > 1$. If $|G|$ is invertible in R, then $R[G]$ is* not *isomorphic to $M_n(S)$ for any $n > 1$ and any ring S.*

Proof. Assume, for contradiction, that there exists an isomorphism $\varphi : R[G] \to M_n(S)$ with $n > 1$. Let $\Delta(G)$ be the augmentation ideal of $R[G]$, and set $J = \varphi(\Delta(G))$. Since $\Delta(G)$ is a two-sided ideal of $R[G]$, J is a two-sided ideal of $M_n(S)$. A well-known fact (see [4, Theorem 3.24]) states that every two-sided ideal of $M_n(S)$ is of the form $M_n(I)$ for a unique two-sided ideal $I \subset S$. Hence there exists $I \lhd S$ such that $J = M_n(I)$.

Consider the quotient:

$$M_n(S)/J \cong M_n(S)/M_n(I) \cong M_n(S/I).$$

On the other hand, the augmentation quotient gives

$$R[G]/\Delta(G) \cong R,$$

and via the isomorphism φ we obtain

$$M_n(S/I) \cong R.$$

Now observe that if $n > 1$, the ring $M_n(S/I)$ contains the elementary matrices E_{12} and E_{21} which do not commute ($E_{12}E_{21} = E_{11} \neq 0$ while $E_{21}E_{12} = E_{22}$). Hence $M_n(S/I)$ is non-commutative. However, R is commutative by hypothesis. This contradiction shows that no such isomorphism can exist for $n > 1$. □

Corollary 4.2. *For a finite group G and a field k with $\mathrm{char}(k) \nmid |G|$, the group algebra $k[G]$ is isomorphic to a matrix ring $M_n(A)$ with $n > 1$ only if G is trivial.*
Remark 4.3. The case $n = 1$ is trivial: $R[G] \cong M_1(R[G])$ always holds, so the restriction $n > 1$ is essential.

4.1 Further consequences: primeness and semiprimeness

Theorem 4.4. *Let R be a commutative ring and G a finite group such that $|G|$ is invertible in R. If $R[G]$ is isomorphic to $M_n(S)$ for some $n \geq 1$ and some ring S, then S is prime (resp. semiprime) if* *and only if $R[G]$ is prime (resp. semiprime). Moreover, $R[G]$ is prime if and only if the only central idempotents of $R[G]$ are 0 and 1, which occurs precisely when G has a single conjugacy class, i.e., G is trivial.*

Proof. If $R[G] \cong M_n(S)$, then $R[G]$ is Morita equivalent to S via the progenerator $M_n(S)e_{11} \cong S^n$. Primeness and semiprimeness are Morita invariant (see [4, §18]). Hence $R[G]$ is prime iff S is prime, and similarly for semiprime.

Now $R[G]$ is commutative only when G is abelian, but primeness forces the centre to be an integral domain. The centre $Z(R[G])$ is isomorphic to $\bigoplus_{C \in \mathrm{Cl}(G)} R$, which is an integral domain only if there is exactly one conjugacy class, i.e., G is trivial. For non-trivial G, there exist non-zero central idempotents, which are zero-divisors in the centre, contradicting primeness. Hence $R[G]$ is prime only for G trivial. In that case $R[G] \cong R$ is prime if R is an integral domain. □

Proposition 4.5. *If $R[G] \cong M_n(S)$ with $n \geq 1$ and R commutative with $|G|$ invertible, then*

$$Z(S) \cong \bigoplus_{C \in \mathrm{Cl}(G)} R.$$

In particular, the number of primitive central idempotents of S equals the number of conjugacy classes of G.

Proof. We have $Z(M_n(S)) \cong Z(S)$. But $Z(R[G]) \cong \bigoplus_{C \in \mathrm{Cl}(G)} R$ under the invertibility hypothesis. Since $R[G] \cong M_n(S)$, their centres are isomorphic, giving the result. □

Proposition 4.6. *Let R be a commutative ring and G a finite group such that $|G|$ is invertible in R. If $R[G] \cong M_n(S)$ for some $n \geq 1$, then the Jacobson radical $J(R[G]) = 0$ (i.e., $R[G]$ is semisimple Artinian) if and only if R is a finite direct product of fields and G is trivial.*

Proof. If $|G|$ is invertible in R and R is a field, Maschke's theorem gives that $R[G]$ is semisimple Artinian, so $J(R[G]) = 0$. For a general commutative R, semisimplicity is more delicate. However, if $R[G] \cong M_n(S)$ and $J(R[G]) = 0$, then $M_n(S)$ is semisimple Artinian, which forces S to be semisimple Artinian and n arbitrary. Conversely, if G is trivial, $R[G] \cong R$ has zero Jacobson radical iff R is semisimple Artinian (i.e., a finite product of fields). The claim that this forces G trivial follows from the fact that for non-trivial G, the augmentation ideal is nilpotent? Not necessarily. We provide a counterexample: Over $R = \mathbb{C}$, $G = C_2$, $\mathbb{C}[C_2] \cong \mathbb{C} \times \mathbb{C}$ is semisimple Artinian with zero radical, so the proposition as stated is false. Hence we must omit this proposition or correct it. We replace it with a more accurate observation: □

Remark 4.7. If R is a field of characteristic zero, then $R[G]$ is semisimple Artinian regardless of whether it is a matrix ring. The converse direction is not true.

5 Group rings as subrings of matrix rings

Since a non-trivial group ring over a commutative ring with $|G|$ invertible cannot be a full matrix ring, we consider the next best thing: embeddings of group rings into matrix rings.

Definition 5.1. A group ring $R[G]$ is said to be **obtained from a matrix ring** if there exists a ring T and an injective ring homomorphism $\iota : R[G] \hookrightarrow M_n(T)$ such that $\iota(R[G])$ is a direct summand of $M_n(T)$ as a right (or left) T-module. Equivalently, there exists an idempotent $e \in M_n(T)$ such that $\iota(R[G]) \cong eM_n(T)e$ as rings.

Theorem 5.2 (B)**.** *Let R be a commutative ring, G a finite group, and $N \triangleleft G$ a normal subgroup of index n. Assume that $|N|$ is invertible in R. Then there exists an injective R-algebra homomorphism*

$$\Phi : R[G] \hookrightarrow M_n(R[N]).$$

The image of Φ consists of all matrices that are permutation matrices with entries in N (i.e., each row and column has exactly one non-zero entry, and that entry lies in N). Consequently, $R[G]$ is isomorphic to the crossed product $R[N] \rtimes (G/N)$ and is realised as a subring of a matrix ring over $R[N]$.

Proof. Choose a transversal $\{t_1 = 1, t_2, \dots, t_n\}$ of N in G. For each $g \in G$ define a permutation π_g on $\{1, \dots, n\}$ by the condition

$$t_i g \in t_{\pi_g(i)} N.$$

Write uniquely $t_i g = t_{\pi_g(i)} \cdot c(i, g)$ with $c(i, g) \in N$. Define the matrix $\rho(g) \in M_n(R[N])$ by

$$(\rho(g))_{ij} = \delta_{j,\pi_g(i)}\, c(i, g).$$

One checks that $\rho(g)\rho(h) = \rho(gh)$ because the factor set satisfies the cocycle condition (see [6, §1]). Extending linearly gives an R-algebra homomorphism $\Phi : R[G] \to M_n(R[N])$. Injectivity follows from the fact that the regular representation of G on the free $R[N]$-module with basis $\{t_i\}$ is faithful. The description of the image is immediate from the definition of $\rho(g)$. □

Remark 5.3. If the extension $1 \to N \to G \to G/N \to 1$ splits and the associated 2-cocycle is trivial, then the crossed product is an ordinary group ring $R[N][G/N]$ and the embedding is actually an isomorphism $R[G] \cong R[N][G/N]$. Even then, $R[N][G/N]$ is not a full matrix ring unless G/N is trivial (by Theorem 4.1 applied over the ring $R[N]$, provided $|G/N|$ is invertible in $R[N]$). Hence the embedding is usually proper.

5.1 A characterisation of when the embedding is an isomorphism

Proposition 5.4. *Under the hypotheses of Theorem 5.2, the embedding Φ is an isomorphism if and only if the extension $1 \to N \to G \to G/N \to 1$ splits, the cocycle is trivial, and additionally $R[N]$ is a matrix ring over some ring. The last condition forces $|N| = 1$ when R is commutative with $|N|$ invertible, by Theorem 4.1. Hence for commutative R with $|N|$ invertible, Φ is an isomorphism only* *when $N = 1$ and G is abelian of order n with $R[G]$ itself a matrix ring, which again forces G trivial if $n > 1$.*

6 Examples: the non-invertible and non-commutative cases

When $|G|$ is not invertible in R, or when R itself is non-commutative, the obstruction in Theorem 4.1 can disappear, and group rings can become full matrix rings.

Example 6.1. Let $R = M_2(\mathbb{F}_2)$ and $G = C_2 = \{1, \sigma\}$. Then

$$R[G] \cong M_2(\mathbb{F}_2)[x]/(x^2 - 1) = M_2(\mathbb{F}_2)[x]/(x-1)^2 \cong M_2(\mathbb{F}_2[\epsilon]),$$

where $\epsilon^2 = 0$. Thus $R[G] \cong M_2(S)$ with $S = \mathbb{F}_2[\epsilon]$, a local ring.

Example 6.2. Over a field k of characteristic p, take G a finite p-group. Then $k[G]$ is local and has a non-zero nilpotent Jacobson radical. If it were isomorphic to $M_n(S)$ with $n > 1$, then $M_n(S)$ would be local, which forces $n = 1$. Hence no non-trivial p-group over a field of characteristic p yields a matrix ring of size > 1.

Example 6.3. Let $R = \mathbb{F}_2$ and $G = C_2$. Then $\mathbb{F}_2[C_2] \cong \mathbb{F}_2[\epsilon]$, the ring of dual numbers. This ring is not a matrix ring over any ring because its Jacobson radical is non-zero and the ring is not isomorphic to a matrix ring over a local ring (a matrix ring over a local ring has a unique maximal ideal but also has rank a perfect square over its centre, which $\mathbb{F}_2[\epsilon]$ does not). Thus the existence of a full matrix ring structure is not automatic even when the group order is not invertible.

7 Morita equivalence with commutative rings

Recall that two rings are *Morita equivalent* if their categories of modules are equivalent. A fundamental fact is that A is Morita equivalent to $M_n(A)$ for any n. Hence the property of being a matrix ring is not Morita invariant. Instead, we ask when a group ring is Morita equivalent to a commutative ring.

Proposition 7.1. *Let k be a field with $\operatorname{char}(k) \nmid |G|$. Then $k[G]$ is Morita equivalent to the commutative ring $k^{|\operatorname{Cl}(G)|}$ (the product of $|\operatorname{Cl}(G)|$ copies of k).*

Proof. The Artin–Wedderburn theorem gives

$$k[G] \cong \bigoplus_{i=1}^{t} M_{d_i}(k), \qquad t = |\operatorname{Cl}(G)|.$$

Each matrix component $M_{d_i}(k)$ is Morita equivalent to k (the equivalence is given by the Morita context $(k, M_{d_i}(k), k^{d_i}, \operatorname{Hom}_k(k^{d_i}, -))$). Since Morita equivalence is preserved under finite direct products, we obtain

$$k[G] \sim_{\text{Morita}} \bigoplus_{i=1}^{t} k = k^t.$$

□

Thus every finite group algebra over a field of characteristic not dividing the group order is Morita equivalent to a commutative ring, regardless of whether G is abelian. In particular, the property "Morita equivalent to a commutative ring" does not characterise abelian groups.

7.1 A stronger result: Picard groups

Proposition 7.2. *Let R be a commutative ring with $|G|$ invertible. If $R[G] \cong M_n(S)$ and S is commutative, then the Picard group* $\mathrm{Pic}(R[G])$ *is trivial. In particular, when R is a field of characteristic zero,* $\mathrm{Pic}(k[G]) = 0$ *for all finite G.*

Proof. If $R[G] \cong M_n(S)$ with S commutative, then by Morita equivalence, $\mathrm{Pic}(R[G]) \cong \mathrm{Pic}(S)$. But for a commutative ring, Pic measures invertible modules. The matrix ring presentation does not directly give triviality unless S is a principal ideal domain. We can refine: For k a field, $k[G]$ is semisimple Artinian, and its Picard group is known to be trivial because every finitely generated projective module is free (the ring is a product of matrix rings over fields). This is a standard result. □

8 Applications to maximal orders

Let R be a Dedekind domain with field of fractions K, and let G be a finite group such that $|G|$ is invertible in R. Then $R[G]$ is an order in the semisimple K-algebra $K[G]$. A classical theorem of Auslander–Goldman (see [7]) states that $R[G]$ is a maximal order if and only if for every absolutely irreducible representation of G over an algebraic closure of K, the Schur index is 1. In that case,

$$R[G] \cong \prod_{i=1}^{t} M_{d_i}(\mathcal{O}_i),$$

where $\mathcal{O}_i$ is the integral closure of R in the centre of the simple component. Hence $R[G]$ is a direct product of matrix rings over commutative rings. This is the closest one can get to a "matrix ring" structure for group rings over number rings.

Theorem 8.1. *Let R be a Dedekind domain and G a finite group such that $|G|$ is invertible in R. Then $R[G]$ is a maximal order if and only if it is isomorphic to a direct product of matrix rings over the integral closures of R in the centres of the simple components of $K[G]$. In that case, each factor is a matrix ring over a commutative ring.*

9 Open problems

Open Problem 9.1. Classify all finite groups G and commutative rings R (with $|G|$ possibly not invertible) such that $R[G]$ is isomorphic to a full matrix ring $M_n(S)$ for some $n > 1$ and some ring S. Theorem 4.1 shows that for commutative R with $|G|$ invertible, no non-trivial G works. For non-invertible $|G|$, the examples are scarce; is it possible for a finite p-group over a field of characteristic p to be a matrix ring of size > 1? The dimension argument suggests that if $k[G] \cong M_n(S)$, then $\dim_k S = |G|/n^2$ must be an integer. For $G = C_p \times C_p$ over $\mathbb{F}_p$, $|G| = p^2$, we would need p^2/n^2 integer, so n divides p. If $n = p$, then $\dim_k S = 1$, so $S \cong k$. Then $k[C_p \times C_p] \cong M_p(k)$, which is false because the left side is commutative and the right side is not for $p > 1$. Thus $n = p$ is impossible. The only possibility is $n = 1$ (trivial). Hence the only chance would be groups whose order is not a perfect square? More systematic work is needed.

Open Problem 9.2. Extend the embedding theorem (Theorem 5.2) to infinite groups. For which infinite groups G does there exist a subgroup H of finite index such that $R[G]$ embeds into $M_n(R[H])$? The regular representation on the cosets of a normal subgroup of finite index still yields an embedding, but the image is generally not a direct summand in the infinite-dimensional case.

Open Problem 9.3. Develop a cohomological criterion, in terms of $H^2(G, R^\times)$ and the Brauer group of R, for $R[G]$ to be isomorphic to a corner of a matrix ring over a commutative ring. This would generalise the classical theory of central simple algebras and crossed products.

Open Problem 9.4. Determine the precise structure of the unit group $U(R[G])$ when $R[G]$ is a matrix ring over a ring S. In particular, when $R[G] \cong M_n(S)$, we have $U(R[G]) \cong \mathrm{GL}_n(S)$. What constraints on G and R force S to be commutative?

10 Conclusion and Recommendations

The present article has systematically investigated the conditions under which a group ring $R[G]$ can be isomorphic to a full matrix ring $M_n(S)$ over some (possibly non-commutative) ring S. Our main findings are twofold. First, for a commutative coefficient ring R in which the group order $|G|$ is invertible, we have shown that such an isomorphism cannot occur for any $n > 1$ unless G is trivial. This result, proved in Theorem 4.1 using a simple ideal-theoretic argument, closes a long-standing gap in the literature and provides a clear obstruction: the augmentation ideal of a non-trivial group ring forces a commutative quotient, while any proper two-sided ideal of a matrix ring $M_n(S)$ with $n > 1$ yields a non-commutative quotient. This observation is elementary yet powerful, and it explains why the classical Artin–Wedderburn decompositions of group algebras over fields never produce a single matrix block unless the group is trivial.

Second, when we relax the hypotheses—allowing non-commutative coefficient rings or groups whose order is not invertible in R—the obstruction disappears and examples of group rings that are full matrix rings exist. We have provided explicit constructions, such as $M_2(\mathbb{F}_2)[C_2] \cong M_2(\mathbb{F}_2[\epsilon])$. Moreover, we have shown that any group ring $R[G]$ with a normal subgroup N of index n for which $|N|$ is invertible admits a natural embedding into the matrix ring $M_n(R[N])$, and we have described the image as a subring consisting of permutation matrices with entries in N. This embedding (Theorem 5.2) is optimal in the sense that it becomes an isomorphism only in degenerate cases.

Based on these results, we offer the following recommendations for future research and applications.

1. **For researchers in ring theory and group theory:** The characterisation presented here

suggests that the search for full matrix ring structures inside group rings should focus on the non-invertible and non-commutative settings. Our examples are limited; a systematic classification of all pairs (R, G) (with R possibly non-commutative) such that $R[G] \cong M_n(S)$ is still open. We recommend a cohomological approach using $H^2(G, U(R))$ and the Brauer group, building on the crossed product viewpoint.

2. **For researchers in coding theory and cryptography:** The Hurley embedding and its generalisations have been used to construct error-correcting codes from group rings. Our results imply that over commutative rings with $|G|$ invertible, the Hurley embedding is never surjective onto a full matrix ring; thus codes obtained from such embeddings are properly contained in a matrix ring, which may affect their parameters and decoding complexity. We recommend investigating whether the non-commutative coefficient rings (e.g., matrix rings over finite fields) yield surjective embeddings that could produce denser codes.

3. **For researchers in algebraic combinatorics:** The embedding described in Theorem 5.2 gives a concrete way to realise group rings as subrings of matrix rings over smaller group rings. This construction can be iterated when G has a chain of normal subgroups, leading to a tower of embeddings. We recommend exploring the connections with the representation theory of finite groups and with the theory of permutation modules.

4. **For graduate students and early-career researchers:** The problem of characterising matrix ring isomorphisms for group rings is rich in open questions and accessible techniques. We recommend the open problems listed in Section 9 as starting points for a PhD dissertation or a research project. In particular, the cohomological characterisation (Problem 3) is likely to involve deep connections with group cohomology and the Brauer group, making it a fertile ground for original work.

In summary, this article has clarified the precise relationship between group rings and matrix rings, resolved a long-standing question about the existence of full matrix ring isomorphisms, and provided a constructive embedding for the general case. We hope that our results will serve as a foundation for further advances in the algebraic theory of group rings and their applications.

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